Realization of four-state qudits using biphotons

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The novel experimental realization of four-level optical quantum systems (ququarts) is presented. We exploit the polarization properties of the frequency non-degenerate biphoton field to obtain such systems. A simple method that does not rely on interferometer is used to generate and measure the sequence of states that can be used in quantum key distribution (QKD) protocol.

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I. Introduction. Recently multi-dimensional (D>2) quantum systems or qudits attracted much attention in context of quantum information and communication. It is partly caused by fundamental aspects of quantum theory since the usage of qudits allows one to violate Bell-type inequalities longer than with two dimensional systems (for the references see the review [1]). Much interest in qudits also comes from the application point, especially from the applied quantum key distribution (QKD). Multilevel systems are proved to be more robust against noise in the transmission channel, although measurement and preparation procedures of such systems seems to be much more technically complicated than in the case of qubits. Different aspects of the security of qudit-based protocols have been analyzed [2, 3]. Lately a proof-of-principle realization of a QKD protocol with entangled qutrits ($D = 3$) [4] and with qudits [5] have been demonstrated. For the last years several elegant experiments have been performed where different kinds of optical qudits were introduced [5, 6, 7, 8, 9, 10, 11, 12]. Recently an experiment that ensured the full control over a polarization qutrit state was demonstrated [12, 13]. However polarization qutrits do not seem to serve as a practical candidate for the multilevel QKD, since it is impossible to achieve a demanded state using only SU2 transformations that are done with linear optical elements. Meantime transformations done by these elements are needed for practical realization of QKD protocols.

In this Letter we present the results of the experimental preparation, transformation, and measurement carried out with polarization based ququarts or quantum systems with dimensionality $D = 4$.

II. Polarization ququarts and their properties. If one considers the two-photon field, generated via spontaneous parametric down-conversion (SPDC) process, then the pure polarization state can be written as a superposition of four basic components:

$$|\Psi\rangle = c_1|H_1, H_2\rangle + c_2|H_1, V_2\rangle + c_3|V_1, H_2\rangle + c_4|V_1, V_2\rangle. \quad (1)$$

Here $c_i = |c_i|e^{i\phi_i}, (i = 1, 2, 3, 4)$ are complex probability amplitudes, $|H_j\rangle \equiv a^\dagger_{\lambda_j}|vac\rangle$, $|V_j\rangle \equiv b^\dagger_{\lambda_j}|vac\rangle$, where $\lambda_j, (j = 1, 2)$ are the central wavelengths of down converted photons. If the down converted photons have an only polarization degree of freedom, then a ququart state (1) converts to a qutrit state i.e. middle terms in (1) become indistinguishable. In order to distinguish between these terms one must be able to distinguish between the down converted photons either in frequency, momentum or detection time. In experiments, described in this paper, we chose the collinear non-degenerate regime of SPDC, so twin photons that form a biphoton were having different frequencies and propagating simultaneously along the same direction. The sum of their frequencies was equal to the frequency of the pump, according to energy conservation. Polarization properties of this state can be described by Stokes parameters, which are defined as the mean values of Stokes operators, averaged over a state (1). Although the description of the light polarization can be introduced only for the quasimonochromatic plane waves, it is possible to use P-quasispin formalism [14] to describe the polarization of arbitrary quantum beams with n modes, frequency or spatial. It is worthy to note that the frequency representation of the ququart (1) is isomorphic to the spatial one, when twin photons have the same frequencies but propagate in different directions [15]. Frequently in the tasks of quantum communication it is convenient to operate with states in a single spatial mode. For the two-frequency and single-spatial mode field, the formal definition of annihilation/creation operators is given by the sum of corresponding operators in each mode. Also we take into account that these operators do commute for different frequency modes. So the Stokes parameters will contain time dependent terms $exp(i(\omega_1-\omega_2)t)$ that describe "beats" of frequency modes and have no connection with the light polarization. However, these terms vanish if one considers the finite detec-

tion time, that allows to classically average these "beats":

$$
\begin{gathered}
\langle S_0\rangle=\langle a_1^\dagger a_1+a_2^\dagger a_2+b_1^\dagger b_1+b_2^\dagger b_2\rangle=2;\\
\langle S_1\rangle=\langle a_1^\dagger a_1+a_2^\dagger a_2-b_1^\dagger b_1-b_2^\dagger b_2\rangle=2(|c_1|^2-|c_4|^2);\\
\langle S_2\rangle=\langle a_1^\dagger b_1+a_2^\dagger b_2+b_1^\dagger a_1+b_2^\dagger a_2\rangle=\\
2Re(c_1^\star(c_2+c_3)+c_4(c_2^\star+c_3^\star));\\
\langle S_3\rangle=\langle a_1^\dagger b_1+a_2^\dagger b_2-b_1^\dagger a_1-b_2^\dagger a_2\rangle=\\
2Im(c_1^\star(c_2+c_3)+c_4(c_2^\star+c_3^\star)).
\end{gathered}
\tag{2}
$$

The polarization degree is given by

$$
P_4=\frac{\sqrt{\sum\limits_{k=1}^{3}\langle S_k^{(1)}+S_k^{(2)}\rangle^2}}{\langle S_0^{(1)}+S_0^{(2)}\rangle}. \tag{3}
$$

This definition of the polarization degree is just generalization of the commonly used classical one. It differs from the definition suggested in [16], where it serves as a witness of the state purity. In the case of polarization-based qutrit states [12, 13], the polarization degree $P_3=\sqrt{|c_1'|^2-|c_3'|^2+2|c_1'^*c_2'+c_2'^*c_3'|^2}$ with $c_1'=c_1, \sqrt{2}c_2'=c_2=c_3, c_3'=c_4$ in (1) was an invariant to unitary polarization transformations. Indeed it is impossible to prepare all demanded pure states, unless one uses interferometric schemes with several nonlinear crystals [12]. In particular there is no way to transform the basic qutrit state $|\Psi_4'\rangle=|V,V\rangle$ with $P=1$ into the state $|\Psi_2'\rangle=|H,V\rangle$ with $P=0$ using retardation plates. However, in the case of polarization ququarts, this quantity is no longer the invariant and can be changed by applying local unitary transformations in each frequency mode. It can be achieved by using dichroic polarization transformers, which act separately on the photons with different frequencies. For example to transform the state $|\Psi_4\rangle=|V_{\lambda_1},V_{\lambda_2}\rangle$ into the state $|\Psi_2\rangle=|H_{\lambda_1},V_{\lambda_2}\rangle$ one needs to use the retardation plate which serves as a half wave plate at λ_1 and as a wave plate at λ_2. The unitary transformation on the state (1) is given by 4×4 matrix that is obtained by a direct product of two 2×2 matrices describing the transformation performed on each photon:

$$
G\equiv\begin{pmatrix}
t_1t_2 & t_1r_2 & r_1t_2 & r_1r_2\\
-t_1r_2^* & t_1t_2^* & -r_1r_2^* & r_1t_2^*\\
-r_1^*t_2 & -r_1^*r_2 & t_1^*t_2 & t_1^*r_2\\
r_1^*r_2^* & -r_1^*t_2^* & -t_1^*r_2^* & t_1^*t_2^*
\end{pmatrix}, \tag{4}
$$

with "transmission" and "reflection" coefficients
$t_i=\cos\delta_i+i\sin\delta_i\cos2\alpha_i, \quad r_i=i\sin\delta_i\sin2\alpha_i,$
$\delta_i=\pi(n_o-n_e)h/\lambda_i$, where δ_i is its optical thickness, h is the geometrical thickness, and α_i is the orientation angle between the optical axis of a wave plate and vertical direction. So for the unitary transformation of the ququart the matrix considered above has the form

$$
G\equiv\begin{pmatrix}
0&0&1&0\\
0&0&0&1\\
1&0&0&0\\
0&1&0&0
\end{pmatrix}. \tag{5}
$$

The same dichroic plate performs direct and reverse unitary transformations between the polarization Bell states: $|\Phi^{+(-)}\rangle$ and $|\Psi^{+(-)}\rangle$, which are represented by ququarts with c_2=c_3=0, c_1=+(-)c_4=$\frac{1}{\sqrt{2}}$ and c_1=c_4=0, c_2=+(-)c_3=$\frac{1}{\sqrt{2}}$ correspondingly. Similar transformations with frequency non-degenerate biphotons have been realized in [17].

In general to prepare an arbitrary ququart state (1) it is necessary to use four nonlinear crystals arranged in such a way that each crystal emits coherently one basic state in the same direction. But in particular cases a reduced set of crystals is quite sufficient to generate specific ququart states which can be used in applications. For example to prepare the polarization Bell states it was sufficient to use two crystals [17, 18]. Moreover even single crystal allows one to prepare the useful subset of ququarts.

III. Polarization ququarts in QKD protocol. The complete QKD protocol with four-dimensional polarization states exploits five mutually unbiased bases with four states in each. In terms of biphoton states the first three bases consist of product polarization states of two photons while the last two bases consist of two-photon entangled states:

$$
\begin{array}{rllll}
I. & |H_1H_2\rangle; & |H_1V_2\rangle; & |V_1H_2\rangle; & |V_1V_2\rangle,\\
II. & |D_1D_2\rangle; & |D_1A_2\rangle; & |A_1D_2\rangle; & |A_1A_2\rangle,\\
III. & |R_1R_2\rangle; & |R_1L_2\rangle; & |L_1R_2\rangle; & |L_1L_2\rangle,\\
IV. & |R_1H_2\rangle+|L_1V_2\rangle; & |R_1H_2\rangle-|L_1V_2\rangle; & &\\
 & |L_1H_2\rangle+|R_1V_2\rangle; & |L_1H_2\rangle-|R_1V_2\rangle, & &\\
V. & |H_1R_2\rangle+|V_1L_2\rangle; & |H_1R_2\rangle-|V_1L_2\rangle; & &\\
 & |H_1L_2\rangle+|V_1R_2\rangle; & |H_1L_2\rangle-|V_1R_2\rangle. & &
\end{array}
\tag{6}
$$

Here H,V,D,A,R,L indicate horizontal, vertical, +45 and -45 linear, right- and left-circular polarization modes correspondingly, and lower indices numerate the frequency modes of two photons. It has been proved [3] that using only first two or three bases is sufficient for the efficient QKD. Exploiting the incomplete set of bases one sacrifices the security but enhances the key generation rate. That is why we will restrict ourselves to the first three bases and skip the rest two. As we will show experimentally, the set of twelve states can be prepared with a single non-linear crystal and local unitary transformations. We also present a measurement scheme that allows to discriminate the states belonging to one basis deterministically thus allowing its implementation in realization of QKD protocol with polarization ququarts.

IVa. Experimental setup. The experimental setup for generation and measurement of ququart states is shown at

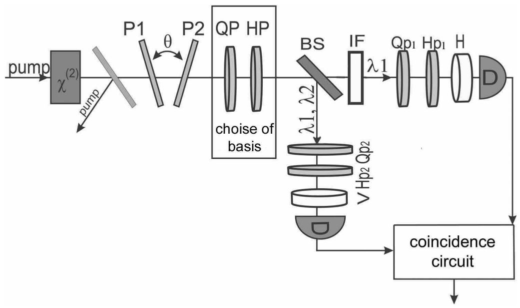

FIG. 1: Setup for preparation and measurement of ququarts

Fig. 1. The $10mW$ cw He-Cd laser operating at $325nm$ serves as a pump. A 15mm lithium-iodate type I crystal emits down-converted photons at central wavelengths $\lambda_1 = 702nm$, $\lambda_2 = 605nm$ within the spectral width of $2nm$ each, propagating collinearly with the pump, so the ququart state $|V_1V_2\rangle$ is generated. Then, this state was subjected to transformations done by dichroic wave plate(s). Finally the state passed through the zero-order half- and quarter plates depending on which bases has been chosen. The measurement setup consists of a Brown-Twiss scheme with a non-polarizing beam-splitter. An interference filter centered at $702nm$ with a FWHM bandwidth of $3nm$ was placed in transmitted arm. This part of the scheme allows to reconstruct arbitrary polarization biphoton-ququart by registering coincidence counts for different projections that are done by the polarization filters located in each arm [19]. Each filter consists of a zero-order quarter- and half-waveplate and a fixed analyzer. Two Si-APD's, linked to a coincidence scheme with 1.5 nsec time window, were used as single photon detectors.

IVb. Experimental procedure. Let us consider as an example, the preparation of a state $|H_1V_2\rangle$ from the initial state $|V_1V_2\rangle$. This transformation can be achieved by a dichroic wave plate oriented at 45° that introduces a phase shift of 2π between extra- and ordinary polarized photons at $605nm$, and a phase shift of π for the conjugate photons at $702nm$. Using quartz as birefringent material it is easy to calculate that the one of possible thicknesses of the wave plate that does this transformation should be equal to $3.406mm$ [20]. We used two plates P1 and P2 with an effective thickness of $3.401mm$. If then one can tilt these wave plates towards each other by a finite angle θ, then the optical thickness of the effective wave plate will be changing and, at a certain value of θ, the desired transformation will be achieved. This corresponds to maximal coincidence rate when the measurement part is tuned to select $|H_1V_2\rangle$ state. Monitoring the coincidences, one can obtain the value of θ for which the main maximum occurs. Then, fixing the tilting angle at this value, one can perform a complete quantum state tomography protocol in order to verify if the state really coincides with the ideal. In order to change the basis from I to II (III), zero order half- (quarter) wave plates oriented at 22.5^0 (45^0) were used.

V. Results and discussion. Fig. 2 shows the coincidences and single count rates versus the change of tilting angle θ which determines optical thickness of the effective wave plate. If the measurement setup is tuned to select the state $|H_1V_2\rangle$ then dependence of coincidences rate on the plate optical thicknesses δ_i is given by formula:

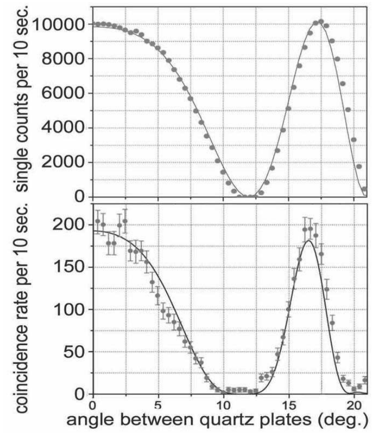

FIG. 2: Dependence of single counts (upper) and coincidences (lower) on tilting angle θ

$$R_{coin} \propto \langle {a_1}^{\dagger} {b_2}^{\dagger} a_1 b_2 \rangle = \sin^2(\delta_1)\cos^2(\delta_2), \qquad (7)$$

whereas the single counts distribution in the upper channel is given by

$$I_{702nm} \propto \langle {a_1}^{\dagger} a_1 \rangle = \sin^2(\delta_1). \qquad (8)$$

The solid lines at the Fig. 2 show the theoretical curves. We performed tomography measurements for the main and additional maxima as well as for the minimum. The minima in coincidences occur when intensity in any channel drops to zero, so it is not a necessary condition for distinguishing the orthogonal state to the one selected by given settings of polarization filters. Nevertheless according to calculations and our measurements the minimum in the coincidences at Fig. 2 exactly refers to the state $|V_1H_2\rangle$. Starting from the $|V_{702nm}V_{605nm}\rangle$ we prepared and measured the whole set of the states from (6) belonging to the first three bases. The table I show components of the experimental (theoretical) density matrix as well as fidelity F defined by $F = Tr(\rho_{th}\rho_{exp})$ for some of the states.

The main obstacle for the practical implementation of the free-space QKD protocol based on ququarts is that one needs to perform a fast polarization transformation at the selected wavelengths. At the same time the method discussed in this Letter allows one to unambiguously distinguish all states forming chosen bases. The measurement set-up which has been already tested

State	$\rho_{11}^{exp}(\rho_{11}^{th})$	$\rho_{22}^{exp}(\rho_{22}^{th})$	$\rho_{33}^{exp}(\rho_{33}^{th})$	$\rho_{44}^{exp}(\rho_{44}^{th})$	F
$\|H_1V_2\rangle$	0(0)	0.984(1)	0(0)	0.016(0)	0.98
$\|V_1H_2\rangle$	0.024(0)	0.004(0)	0.944(0)	0.028(0)	0.94
$\|D_1A_2\rangle$	0(0)	0.994(1)	0(0)	0.006(0)	0.99
$\|A_1D_2\rangle$	0.015(0)	0(0)	0.950(1)	0.035(0)	0.95
$\|R_1L_2\rangle$	0(0)	0.973(1)	0.027(0)	0(0)	0.97
$\|L_1R_2\rangle$	0.016(0)	0(0)	0.957(1)	0.027(0)	0.96

TABLE I: Density matrix components and fidelities for selected transformations.

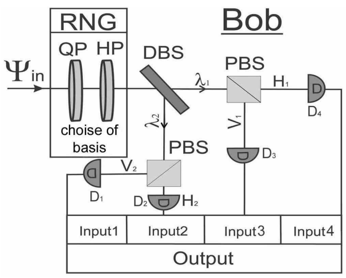

FIG. 3: Measurement part at Bob's station

in our experiments is shown on the Fig. 3. It consists of the dichroic mirror, separating the photons with the different wavelenghts, and a pair of polarization beam-splitters, separating photons with the orthogonal polarizations. We would like to stress that using dichroic beam-splitter allows one to achieve 100% mode separation efficiency which is not possible for qutrits. Four-input double-coincidence scheme linked to the outputs of single-photon detectors registers the biphotons-ququarts. For example for the first basis, the scheme works as follows, provided that Bob's guess of the basis is correct:

if state $|H_1H_2\rangle$ comes, then detectors D4, D2 will fire,

if state $|H_1V_2\rangle$ comes, then detectors D4, D1 will fire,

if state $|V_1H_2\rangle$ comes, then detectors D3, D2 will fire,

if state $|V_1V_2\rangle$ comes, then detectorsD3, D1 will fire.

Same holds for any of the remaining correctly guessed bases, since the quarter- and half wave plates transform the polarization to HV basis in which the measurement is performed. Registered coincidence count from a certain pair of detectors contributes to corresponding diagonal component of the measured density matrix. So if the basis is guessed correctly, then the registered coincidence count deterministically identifies the input state. We illustrate this statement by the table which shows total number of registered events per 30 sec for the input state $|R_1L_2\rangle$ measured in circular basis and calculated components of the experimental (theoretical) density matrix. Moreover, registering coincidences allows one to circumvent the problem of the detection noise that is common for single-photon based protocols. If the coincidence window is quite small, it is possible to assure a very low level of accidental coincidences for the usual dark count rate of single photon detectors.

D_4D_2	ρ_{11}	D_4D_1	ρ_{22}	D_3D_2	ρ_{33}	D_3D_1	ρ_{44}
0	0.0(0)	220	0.973(1)	6	0.027(0)	0	0.0(0)

TABLE II: Coincidence rate and density matrix components

To conclude we have suggested and tested a novel method of the preparation, and measurement of the subset of four-dimensional polarization quantum states. Since for this class of states the polarization degree is not invariant under SU2 transformations it is possible to switch between states using simple polarization elements.

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transformation is determined by the number of order m_i that is chosen for the specified wavelength according to $\frac{m_2 \lambda 2}{\delta n_2} = \frac{(2m_1+1)\lambda 1}{2\delta n_1}$. Here δn_i is the value of the birefringence for the specific wavelength. We chose order $m_1 = 51$ for $\lambda = 702nm$ and $m_2 = 47$ for $\lambda = 650nm$.